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# On a superfield extension of the ADHM construction and $\mathcal{N} = 1$ super instantons

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Abstract: We give a superfield extension of the ADHM construction for the euclidean theory obtained by Wick rotation from the lorentzian four dimensional $\mathcal{N} = 1$ super Yang-Mills theory. In particular, we investigate the procedure to guarantee the Wess-Zumino gauge for the superfields obtained by the extended ADHM construction, and show that the known super instanton configurations are correctly obtained.

Keywords: Solitons Monopoles and Instantons, Gauge Symmetry, Superspace.

## Contents





## 1. Introduction

Instantons and their effects in field theory have been investigated for more than two decades (see, for example, [1, 2] and references therein) and still attract much attention in theoretical physics. It is well known that the instanton configurations of the gauge field can be obtained by the ADHM construction [3]. In supersymmetric theories, there are zero modes of adjoint fermions in the instanton background, which naturally introduce the superpartner of the bosonic moduli called Grassmann collective coordinates (or fermionic moduli). The fermion zero modes together with the bosonic configurations are called super instantons. To give the super instanton solutions, superfield extensions of the ADHM construction were proposed in [4, 5], in which the fermionic moduli belong to the same superfield containing the corresponding bosonic moduli (see also [6]–[11] for related works). Recently it was found that supersymmetric gauge theory defined in a kind of deformed superspace, called non(anti)commutative superspace, arises in superstring theory as a low energy effective theory on D-branes [12]. To obtain instantons in such a deformed theory the superfield extension of the ADHM construction would be useful, because such a non(anti)commutative theory is realized by deforming the multiplication of superfields.

It is recognized, however, that even without deformation there exists a gap between the superfield extension of the ADHM construction and the super instantons obtained by the component formalism. We will consider the euclidean theory obtained by Wick rotation from the lorentzian four dimensional $\mathcal{N} = 1$ super Yang-Mills (SYM) theory.

Hereafter, "$\mathcal{N}=1$" stands for the counting of the supercharges in the lorentzian theory before Wick rotation. The ADHM construction gives the instanton gauge fields. One of the fundamental objects in the ADHM construction is the "zero-dimensional Dirac operator" $\Delta_\alpha(x) = a_\alpha + x_{\alpha\dot{\alpha}}b^{\dot{\alpha}}$, where $a$ contains the bosonic moduli parameters. Given the zero modes $v$ of $\Delta_\alpha$, the instanton gauge fields are constructed and written in terms of $a$ and $v$. In $\mathcal{N}=1$ SYM theory, there exist fermion zero modes coming from the gaugino in the instanton backgrounds. It is well known that they are expressed by using the ADHM data and the fermionic moduli parameters $\mathcal{M}$. When we consider an $\mathcal{N}=1$ superfield extension of the ADHM construction, the operator $\Delta_\alpha(x)$ defined above is extended to a chiral superfield $\hat{\Delta}_\alpha(y,\theta) = \Delta_\alpha(y) + \theta_\alpha\mathcal{M}$ as was proposed in [4, 5]. A superfield obtained from the extended ADHM construction as a super instanton configuration should be taken in the Wess-Zumino (WZ) gauge in order to be compared with the known results from the component formalism and, of course, should be consistent with them in its higher components as a superfield. So far there is no elaborated procedure to guarantee the WZ gauge for the superfields obtained by the extended ADHM construction.

It has been also argued [13] that superfield extensions of the ADHM construction for four dimensional euclidean theories exist only for theories with extended supersymmetry or complexified gauge group. In formulating the extended ADHM construction, the authors of refs. [4, 5] have introduced a counterpart of $\hat{\Delta}_\alpha(y)$.[1] In order to include the purely bosonic ADHM construction, this counterpart should coincide with the complex conjugate of $\Delta_\alpha(x)$ when $\theta, \bar{\theta} \to 0$. If we assume that the counterpart is obtained by a kind of conjugation operation, such a conjugation maps a chiral (antichiral) superfield in $\mathcal{N}=1$ theory to another chiral (antichiral) superfield. In particular, this implies that the undotted (dotted) spinor $\theta^\alpha$ ($\bar{\theta}_{\dot{\alpha}}$) is self-conjugate under this conjugation. Since Majorana spinors do not exist in four dimensional euclidean space, the Grassmann coordinates $\theta^\alpha$ and $\bar{\theta}_{\dot{\alpha}}$ in $\mathcal{N}=1$ theory are necessarily (independent) complex spinors[2] and we cannot impose the desired reality condition on $\theta$ and $\bar{\theta}$ with respect to the complex conjugation.

In this paper, we would like to establish an $\mathcal{N}=1$ extended ADHM construction, i.e. a superfield extension of the ADHM construction for the euclidean theory obtained by Wick rotation from the lorentzian four dimensional $\mathcal{N}=1$ SYM theory. We consider SU($n$) or U($n$) gauge groups for definiteness. As was pointed out in ref. [16], for example, it is useful to introduce a kind of conjugation (which we denote by "‡") in order to analyze the euclidean version of the $\mathcal{N}=1$ theory. In terms of the ‡-conjugation, we are allowed to impose "reality" conditions for spinors: For example, $(\theta_\alpha)^\ddagger = \varepsilon^{\alpha\beta}\theta_\beta$, $(\bar{\theta}_{\dot{\alpha}})^\ddagger = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}$. These conditions imply that $^\ddagger$ squares to $-1$ on spinors and this will be one of the characteristic properties of the conjugation. With the use of the ‡-conjugation, we investigate the extended ADHM construction, where especially care is taken to ensure the WZ gauge for the superfields from the extended ADHM construction. We show that the ADHM construction

[1] In ref. [5], it is the transposed matrix of $\hat{\Delta}_\alpha(y)$.

[2] Similarly $\lambda^\alpha$ and $\bar{\lambda}_{\dot{\alpha}}$ also become independent complex spinors. Nevertheless, in the path integral, Grassmann integration $\mathcal{D}\lambda$ does not distinguish real and complex spinors [14, 15], so that the fact that $\lambda^\alpha$ and $\bar{\lambda}_{\dot{\alpha}}$ are independent complex spinors does not matter as far as we are interested in the instanton contribution to the path integral of the lorentzian four dimensional $\mathcal{N}=1$ theory [1].

can be consistently extended in the $\mathcal{N}=1$ superfield formalism. As a result we will see that the obtained gauge potential can be real, allowing us to choose SU($n$) or U($n$) gauge groups rather than their complexification.

This paper is organized as follows. In section 2, we review the geometric construction of the euclidean version of $\mathcal{N}=1$ SYM theory and the $\mathcal{N}=1$ extended ADHM construction proposed in [4, 5]. In section 3, we find the conditions in the $\mathcal{N}=1$ extended ADHM construction to give superfields in the WZ gauge and show that the resulting superfields are actually consistent with known results. Section 4 is devoted to conclusions and discussion. Our notation and conventions are summarized in appendix A. Note that we are working on four dimensional euclidean space but we will use the lorentzian signature notation of [17]. Finally, in appendix B, we describe the ‡-conjugation rules which we have adopted.

## 2. The super ADHM construction

### 2.1 The geometric construction of $\mathcal{N}=1$ super Yang-Mills

In this subsection, we review the geometric construction of the euclidean version of the $\mathcal{N}=1$ SYM theory obtained by Wick rotation (see [17, 19, 20] and appendix A for our notation and conventions), using the superspace formalism. In this construction the basic object is the connection superfield $\phi_A=(\phi_\mu,\phi_\alpha,\phi^{\dot\alpha})$ on the superspace, and the curvature superfield $F_{AB}$ is defined (see eq. (A.18) in appendix A). $F_{AB}$ satisfies the Bianchi identities (A.20).

The curvature $F_{AB}$ contains many redundant fields and the following constraints are known to be the proper ones to get rid of the redundant fields and to reproduce the multiplet of $\mathcal{N}=1$ SYM theory:

$$F_{\alpha\beta}=F_{\dot\alpha\dot\beta}=F_{\alpha\dot\beta}=0\,. \tag{2.1}$$

We choose the following solution to these constraints:

$$\phi_\alpha=-e^{-V}D_\alpha e^V\,,\qquad \phi^{\dot\alpha}=0\,,\qquad \phi_\mu=-\frac{i}{4}\bar\sigma_\mu^{\dot\beta\beta}\bar D_{\dot\beta}\phi_\beta\,. \tag{2.2}$$

where $V=V^rT^r$, $V^r$ are superfields and $T^r$ denote the hermitian generators of SU($n$) or U($n$). Then the non-trivial curvature superfields satisfying the Bianchi identities are written as

$$F_{\mu\dot\alpha}=\frac{i}{2}\mathcal{W}^\beta\sigma_{\mu\beta\dot\alpha}\,,\qquad F_{\mu\alpha}=\frac{i}{2}\sigma_{\mu\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta}\,,\qquad F_{\mu\nu}=-\frac{1}{4}(\bar{\mathcal{D}}\bar\sigma_{\mu\nu}\bar{\mathcal{W}}-\mathcal{D}\sigma_{\mu\nu}\mathcal{W})\,, \tag{2.3}$$

where

$$\mathcal{W}^\alpha=W^\alpha\,,\qquad \bar{\mathcal{W}}_{\dot\alpha}=e^{-V}\bar W_{\dot\alpha}e^V\,, \tag{2.4}$$

and

$$W_\alpha=-\frac{1}{4}\bar D_{\dot\beta}\bar D^{\dot\beta}\left(e^{-V}D_\alpha e^V\right),\qquad \bar W_{\dot\alpha}=\frac{1}{4}D^\beta D_\beta\left(e^V\bar D_{\dot\alpha}e^{-V}\right). \tag{2.5}$$

Even after fixing the solution such that $\phi_{\dot\alpha}=0$, there remains the following gauge freedom:

$$e^V\mapsto e^{-i\bar\Lambda'}e^Ve^{i\Lambda}\,, \tag{2.6}$$



JHEP08(2005)065

where $\Lambda$ and $\bar{\Lambda}'$ are an independent chiral and anti-chiral superfield in the adjoint representation, respectively. This remaining gauge freedom allows us to bring the superfield $V$ into the following form (the WZ gauge):

$$V = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) . \tag{2.7}$$

Here all the component fields are complex. In this gauge, the remaining gauge freedom of (2.6) is the one with $\Lambda(y,\theta) = \varphi(y)$ and $\bar{\Lambda}'(\bar{y},\bar{\theta}) = \varphi'(\bar{y})$ where $\varphi$ and $\varphi'$ are complex valued functions. We require $\varphi'(x) = \varphi(x)$ which implies that the transformation (2.6) gives the ordinary gauge transformation laws for the component fields. Hereafter, we call $V$ a vector superfield. In the WZ gauge, we have

$$\phi_\alpha = \left[(\sigma^\mu\bar{\theta})_\alpha(v_\mu + i\theta\sigma_\mu\bar{\lambda}) - \bar{\theta}\bar{\theta}W_\alpha\right](y), \tag{2.8}$$

$$\phi^{\dot{\alpha}} = 0 , \tag{2.9}$$

$$\phi_\mu = -\frac{i}{2}\left[v_\mu + i\theta\sigma_\mu\bar{\lambda} - \bar{\theta}\bar{\sigma}_\mu W\right](y) \tag{2.10}$$

and

$$W_\alpha = -i\lambda_\alpha(y) + \theta^\gamma\Big\{\varepsilon_{\alpha\gamma}D - i(\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma}v_{\mu\nu}\Big\}(y) + \theta\theta(\sigma^\mu\mathcal{D}_\mu\bar{\lambda})_\alpha(y) , \tag{2.11}$$

$$\bar{W}_{\dot{\alpha}} = i\bar{\lambda}_{\dot{\alpha}}(\bar{y}) + \bar{\theta}_{\dot{\gamma}}\Big\{\delta^{\dot{\gamma}}_{\dot{\alpha}}D - i(\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}{}_{\dot{\alpha}}v_{\mu\nu}\Big\}(\bar{y}) + \bar{\theta}\bar{\theta}(\mathcal{D}_\mu\lambda\sigma^\mu)_{\dot{\alpha}}(\bar{y}) . \tag{2.12}$$

Here $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + \frac{i}{2}[v_\mu, v_\nu]$ and $\mathcal{D}_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + \frac{i}{2}[v_\mu, \bar{\lambda}]$.

The lagrangian is given by

$$\mathcal{L} = \frac{-1}{4g^2}\left(\int d^2\theta \mathrm{tr} W^\alpha W_\alpha + \int d^2\bar{\theta}\mathrm{tr}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right) = \frac{-1}{g^2}\mathrm{tr}\left[-\frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda + \frac{1}{2}D^2\right] . \tag{2.13}$$

### 2.2 The $\mathcal{N} = 1$ extended ADHM construction

Throughout this paper, we concentrate on anti-self dual (ASD) gauge fields. The super ASD instanton configurations satisfy the following equations:

$$\star v_{\mu\nu} = -v_{\mu\nu} , \qquad \sigma^\mu\mathcal{D}_\mu\bar{\lambda} = 0 , \qquad \lambda_\alpha = 0 , \qquad D = 0 , \tag{2.14}$$

where $\star$ is the Hodge star: $\star v^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}v_{\rho\sigma}$ (see appendix A for our convention).

Let us first briefly review the ordinary ADHM construction [3] with SU($n$) (or U($n$)) gauge group. The ADHM construction gives the ASD field strength. Let us consider a $k$ instanton configuration. Define $\Delta_\alpha(x)$ such as

$$\Delta_\alpha(x) = a_\alpha + x_{\alpha\dot{\alpha}}b^{\dot{\alpha}} \tag{2.15}$$

where $a_\alpha$ and $b^{\dot{\alpha}}$ are constant $k \times (n + 2k)$ matrices and $x_{\alpha\dot{\alpha}} \equiv ix_\mu\sigma^\mu_{\alpha\dot{\alpha}}$.[3] We assume that $\Delta_\alpha$ has maximal rank everywhere except for a finite set of points. Its hermitian conjugate $\Delta^{\dagger\alpha} \equiv (\hat{\Delta}_\alpha)^\dagger$ is given by

$$\Delta^{\dagger\alpha}(x) = a^{\dagger\alpha} + b^\dagger_{\dot{\beta}}x^{\dot{\beta}\alpha} . \tag{2.16}$$

[3]We include the factor $i$ in the definition of $x_{\alpha\dot{\alpha}}$ to ensure $(x_{\alpha\dot{\alpha}})^* = +x^{\dot{\alpha}\alpha}$, where "$*$" denotes the complex conjugate. See appendix A.

Then the gauge field $v_\mu$ is given by

$$v_\mu = -2iv^\dagger \partial_\mu v\,, \tag{2.17}$$

where $v$ is the set of the normalized zero modes of $\Delta_\alpha$:

$$\Delta_\alpha v = 0\,, \qquad v^\dagger v = \mathbf{1}_n\,. \tag{2.18}$$

After imposing the bosonic ADHM constraints

$$\Delta_\alpha \Delta^{\dagger\beta} \propto \delta_\alpha^\beta\,, \tag{2.19}$$

$a_\alpha$ can be identified with the bosonic instanton moduli parameters. Finally we find the ASD field strength which is implicitly written in terms of the bosonic moduli:

$$v_{\mu\nu} = 8iv^\dagger b^\dagger \bar\sigma_{\mu\nu} f b v \tag{2.20}$$

where $f$ is defined as the inverse of

$$f^{-1} = \frac{1}{2}\Delta_\alpha \Delta^{\dagger\alpha}\,. \tag{2.21}$$

In $\mathcal{N}=1$ SYM theory, there exist zero modes of the adjoint fermion $\bar\lambda_{\dot\alpha}$ in the ASD instanton background (while $\lambda^\alpha = 0$) and they are written with the use of the ADHM data [18] as

$$\bar\lambda_{\dot\alpha} = 2iv^\dagger (b^\dagger_{\dot\alpha} f \mathcal{M} - \mathcal{M}^\dagger f b_{\dot\alpha}) v\,, \tag{2.22}$$

where $\mathcal{M}$ is a $k\times(n+2k)$ Grassmann odd matrix satisfying the fermionic ADHM constraints

$$\mathcal{M}\Delta^\dagger_\alpha + \Delta_\alpha \mathcal{M}^\dagger = 0\,. \tag{2.23}$$

$\mathcal{M}$ can be identified with the fermionic moduli parameters after imposing the fermionic ADHM constraints.

The super instanton condition (2.14) can be rewrite in the superfield formalism [4, 5] as

$$F_{\mu\dot\alpha} = 0\,, \tag{2.24}$$

$$\star F_{\mu\nu} = -F_{\mu\nu}\,, \tag{2.25}$$

where $F$ is the curvature superfield satisfying the Bianchi identity. Actually, substituting eqs. (2.14) into $W_\alpha$ in (2.11), we find $W_\alpha = 0$. This is equivalent to $\mathcal{W}_\alpha = 0$ as seen from (2.4). Then (2.24) and (2.25) follow from eq. (2.3). The equation (2.24) is called the super ASD condition [5] (note that eq. (2.24) implies eq. (2.25)). For later use, we give the expression of $F_{\mu\nu}$ for the super instanton configuration:

$$F_{\mu\nu} = -\frac{i}{2}\left(v^{\mathrm{ASD}}_{\mu\nu} - i\theta\sigma^\rho\bar\sigma_{\mu\nu}\mathcal{D}_\rho\bar\lambda - i\theta\theta\bar\lambda\bar\sigma_{\mu\nu}\bar\lambda\right), \tag{2.26}$$

where $v^{\mathrm{ASD}}_{\mu\nu}$ is the ASD part of the field strength $v_{\mu\nu}$. Note also that the connection superfields (2.8)–(2.10) for the super instantons take the following forms:

$$\phi_\alpha = (\sigma^\mu\bar\theta)_\alpha (v_\mu + i\theta\sigma_\mu\bar\lambda)(y)\,, \qquad \phi_{\dot\alpha} = 0\,, \qquad \phi_\mu = -\frac{i}{2}(v_\mu + i\theta\sigma_\mu\bar\lambda)(y)\,. \tag{2.27}$$

To solve the super ASD condition (2.24), we introduce a superfield extension of the ADHM construction along the line of refs. [4, 5]. By construction, as we will see below, the solutions of the extended ADHM construction include all the solutions given in the component formalism. We define the following quantity which might be interpreted as a superfield extension of the zero dimensional Dirac operator in the WZ gauge:

$$\hat{\Delta}_\alpha = \Delta_\alpha(y) + \theta_\alpha \mathcal{M} \,, \tag{2.28}$$

where $\Delta_\alpha(y)$ is the zero dimensional Dirac operator in the ordinary ADHM construction with replacing $x^\mu$ by the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\mathcal{M}$ is a $k \times (n+2k)$ fermionic matrix which includes the fermionic moduli. We suppose that $\hat{\Delta}_\alpha$ has a maximal rank almost everywhere as in the ordinary ADHM construction. Also, its counterpart $\tilde{\hat{\Delta}}^\alpha$ is introduced by[4]

$$\tilde{\hat{\Delta}}^\alpha \equiv \Delta^{\dagger\alpha}(y) + \theta^\alpha \mathcal{M}^\dagger \,. \tag{2.29}$$

We will show that $\tilde{\hat{\Delta}}^\alpha$ can be identified with the ‡-conjugation of $\hat{\Delta}_\alpha$ in the next section.

The operator $\hat{\Delta}_\alpha$ defined in eq. (2.28) is a chiral superfield without the highest component. This can be explained in the following way. Since the operator $\Delta_\alpha$ appearing in the ordinary ADHM construction is related with the Dirac operator reduced to zero dimensional space, we expect that its superfield extension $\hat{\Delta}_\alpha$ is related to the reduction of the super connection $\phi_\mu$. In fact, we see from eq. (2.27) that the reduction of the connection $\phi_\mu$ for the super instantons actually gives a chiral superfield without the highest component in the WZ gauge. The linearity of $\hat{\Delta}_\alpha$ in terms of $y^\mu$ and $\theta_\alpha$ is also suggested by the supersymmetric version of twistor theory [4] (see also [21, 22, 5]).

Another basic object in the ADHM construction is the zero modes of $\hat{\Delta}_\alpha$. As $\hat{\Delta}_\alpha$ has $n$ zero modes we collect them in a matrix form:

$$\hat{\Delta}_\alpha \hat{v} = 0 \tag{2.30}$$

where $\hat{v}$ is an $(n+2k) \times n$ matrix of superfields. The zero mode superfield $\tilde{\hat{v}}$ of $\tilde{\hat{\Delta}}^\alpha$ is also defined by

$$\tilde{\hat{v}} \tilde{\hat{\Delta}}^\alpha = 0 \,, \tag{2.31}$$

which is an $n \times (n+2k)$ matrix. By construction its lowest component is $v^\dagger$[5] We require $\hat{v}$ and $\tilde{\hat{v}}$ to satisfy

$$\tilde{\hat{v}} \hat{v} = 1 \,. \tag{2.32}$$

The matrix $\hat{v}$ defines an embedding of the $n$-dimensional vector space into the $(n+2k)$-dimensional vector space. And the projection operator $\hat{v}\tilde{\hat{v}}$ defines the $n$ dimensional vector bundle embedded in the $n+2k$ dimensional trivial vector bundle. Using $\hat{v}$ we can pullback

[4]In ref. [5], the counterpart of $\hat{\Delta}_\alpha$ is defined as the transposed matrix of $\hat{\Delta}_\alpha$.

[5]Note that in ref. [5] the zero mode superfield $\tilde{\hat{v}}$ of $\tilde{\hat{\Delta}}^\alpha$ is given by the transposed matrix of $\hat{v}$, because $\tilde{\hat{\Delta}}^\alpha$ is defined as the transposed matrix of $\hat{\Delta}_\alpha$. It results in complex field strengths, as we will be able to see from $F_{\mu\nu}$ in (2.40) whose lowest component is $-\frac{i}{2}v_{\mu\nu}$. In this case, the field strength reads $v_{\mu\nu} = 8iv^{\mathrm{T}}b^{\mathrm{T}}\bar{\sigma}_{\mu\nu}fbv$ where $\hat{f}^{-1} \equiv \frac{1}{2}\hat{\Delta}_\alpha\hat{\Delta}^{\mathrm{T}\alpha}$.

the trivial connection of the trivial bundle with rank $(n+2k)$, and get the non-trivial connection of the vector bundle with rank $n$ as follows:

$$\phi = -\widetilde{\hat{v}} d\hat{v} \,. \tag{2.33}$$

where $d$ is exterior derivative of superspace (see appendix A). The connection $\phi$ defines the curvature

$$F = d\phi + \phi\phi = \widetilde{\hat{v}} d\widetilde{\hat{\Delta}}^{\alpha} \hat{K}_{\alpha}{}^{\beta} d\hat{\Delta}_{\beta} \hat{v} \,, \tag{2.34}$$

where

$$\hat{K}^{-1}{}_{\alpha}{}^{\beta} \equiv \hat{\Delta}_{\alpha} \widetilde{\hat{\Delta}}^{\beta} = \Delta_{\alpha}\Delta^{\dagger\beta} + \theta^{\gamma}\Big(\Delta_{\alpha}\delta_{\gamma}^{\beta}\mathcal{M}^{\dagger} + \varepsilon_{\alpha\gamma}\mathcal{M}\Delta^{\dagger\beta}\Big) + \theta\theta\Big(\frac{1}{2}\delta_{\alpha}^{\beta}\mathcal{M}\mathcal{M}^{\dagger}\Big) \tag{2.35}$$

and $\hat{K}_{\alpha}{}^{\beta}$ is defined such that $\hat{K}^{-1}{}_{\alpha}{}^{\beta}\hat{K}_{\beta}{}^{\gamma} = \hat{K}_{\alpha}{}^{\beta}\hat{K}^{-1}{}_{\beta}{}^{\gamma} = \delta_{\alpha}^{\gamma}\mathbf{1}_k$. To obtain the expression (2.34), we have used the completeness condition:

$$\hat{v}\widetilde{\hat{v}} = \mathbf{1}_{n+2k} - \widetilde{\hat{\Delta}}^{\alpha}\hat{K}_{\alpha}{}^{\beta}\hat{\Delta}_{\beta} \,. \tag{2.36}$$

The coefficient of the 2-form $F$ defines the superfield $F_{AB}$:

$$F_{AB} = -\widetilde{\hat{v}} D_{[A}\widetilde{\hat{\Delta}}^{\alpha}\hat{K}_{\alpha}{}^{\beta} D_{B\}}\hat{\Delta}_{\beta}\hat{v} \,. \tag{2.37}$$

If $\hat{K}$ commutes with the Pauli matrix, the field strength $F_{\mu\nu}$ becomes ASD as in the bosonic ADHM construction although it is a superfield now. This is equivalent to the condition $\hat{\Delta}_{\alpha}\widetilde{\hat{\Delta}}^{\beta} \propto \delta_{\alpha}^{\beta}$ and thus

$$\hat{K}^{-1}{}_{\alpha}{}^{\beta} = \delta_{\alpha}^{\beta}\hat{f}^{-1} \tag{2.38}$$

where

$$\hat{f}^{-1} \equiv \frac{1}{2}\hat{\Delta}_{\alpha}\widetilde{\hat{\Delta}}^{\alpha} \tag{2.39}$$

is a $k\times k$ matrix superfield. There exists $\hat{f}$ because we have assumed that $\hat{\Delta}_{\alpha}$ has maximal rank. The above condition (2.38) leads to both the bosonic ADHM constraints (2.19) and the fermionic ones (2.23). When eq. (2.38) holds, i.e., the parameters in $\hat{\Delta}_{\alpha}$ are satisfying both bosonic and fermionic ADHM constraints, we obtain from eq. (2.37) the ASD curvature superfield (2.25) and another (non-trivial) one in terms of the ADHM quantities:

$$F_{\mu\nu} = 4\widetilde{\hat{v}} b^{\dagger}\bar{\sigma}_{\mu\nu}\hat{f} b\hat{v} \,, \tag{2.40}$$

$$F_{\mu\alpha} = \frac{i}{2}\sigma_{\mu\alpha\dot{\beta}}\Big\{-2\widetilde{\hat{v}}(b^{\dagger\dot{\beta}}\hat{f}\mathcal{M} - \mathcal{M}^{\dagger}\hat{f}b^{\dot{\beta}})\hat{v} - 8\bar{\theta}_{\dot{\gamma}}\widetilde{\hat{v}}(b^{\dagger\dot{\beta}}\hat{f}b^{\dot{\gamma}} + b^{\dagger\dot{\gamma}}\hat{f}b^{\dot{\beta}})\hat{v}\Big\} \,. \tag{2.41}$$

We can check that the curvature superfields satisfy the covariant constraints (2.1) and the super ASD condition (2.24). $F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0$ and $F_{\mu\dot{\alpha}} = 0$ are simply a consequence of the definition of $\hat{\Delta}_{\alpha}$ as a chiral extension of $\Delta_{\alpha}$. Note that $F_{\alpha\beta} = 0$ is checked with the use of the constraint (2.38) and

$$D_{\beta}\hat{\Delta}_{\alpha} = \varepsilon_{\alpha\beta}(\mathcal{M} + 4\bar{\theta}_{\dot{\beta}}b^{\dot{\beta}}) \,, \qquad D_{\beta}\widetilde{\hat{\Delta}}^{\alpha} = \delta_{\beta}^{\alpha}(\mathcal{M}^{\dagger} + 4b^{\dagger}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}}) \,, \tag{2.42}$$

and the fact that $F_{\alpha\beta}$ is symmetric with respect to $\alpha$ and $\beta$. This implies that all the constraints and the super ASD condition are satisfied by the connection (2.33), with $\hat{v}$ being the zero modes of $\hat{\Delta}_\alpha$ obeying the constraint (2.38).

So far we have reviewed an $\mathcal{N}=1$ generalization of the ADHM construction to the superfield formalism and have shown that the extended ADHM construction formally gives the ASD connection $\phi$ in the superfield formalism. The component fields of super instantons in the WZ gauge can be found without knowing the higher components of $\hat{v}$ and $\hat{f}$. They can be obtained by using the following relations:

$$v_{\mu\nu} = 2iF_{\mu\nu}| \,, \qquad \lambda_\alpha = iW_\alpha| \,, \qquad \bar{\lambda}_{\dot{\alpha}} = -i\bar{\mathcal{W}}_{\dot{\alpha}}| \,, \qquad D = -\frac{1}{2}D^\alpha W_\alpha| \,, \tag{2.43}$$

where $|$ indicates that we take $\theta = \bar{\theta} = 0$ (see also [7]). We find that $\bar{\mathcal{W}}$ is given by

$$\bar{\mathcal{W}}^{\dot{\alpha}} = -2\widetilde{\hat{v}}(b^{\dagger\dot{\alpha}}\hat{f}\mathcal{M} - \mathcal{M}^\dagger\hat{f}b^{\dot{\alpha}})\hat{v} - 8\bar{\theta}_{\dot{\gamma}}\widetilde{\hat{v}}(b^{\dagger\dot{\alpha}}\hat{f}b^{\dot{\gamma}} + b^{\dagger\dot{\gamma}}\hat{f}b^{\dot{\alpha}})\hat{v} \tag{2.44}$$

since $\bar{\mathcal{W}}^{\dot{\alpha}} = \frac{i}{2}\bar{\sigma}^{\mu\dot{\alpha}\beta}F_{\mu\beta}$, and $W^\alpha = 0$ from $\mathcal{W}^\alpha = \frac{i}{2}\bar{\sigma}^{\mu\dot{\beta}\alpha}F_{\mu\dot{\beta}}$. $F_{\mu\nu}$ is given in (2.40). Thus, we obtain

$$v_{\mu\nu} = 8iv^\dagger b^\dagger \bar{\sigma}_{\mu\nu} f b v \,, \tag{2.45}$$

$$\bar{\lambda}_{\dot{\alpha}} = 2iv^\dagger(b^\dagger_{\dot{\alpha}} f\mathcal{M} - \mathcal{M}^\dagger f b_{\dot{\alpha}})v \,, \tag{2.46}$$

$$\lambda_\alpha = D = 0 \,, \tag{2.47}$$

where $f$ is the lowest component of $\hat{f}$ and given by the inverse of (2.21) for the case under consideration. The above fields precisely coincide with the super instanton configuration in eqs. (2.20) and (2.22). Note that here we have only used the lowest components of $\hat{v}$ and $\hat{f}$, hence we do not need to solve the equation (2.30). However, we will solve eq. (2.30) in the next section and determine the full form of the connection $\phi_\mu$, in order to confirm that this formalism can lead to the correct superfield extension of the ordinary ADHM construction. For this end, the information of the higher components of $\hat{v}$ and $\hat{f}$ is necessary and this will be a non-trivial check of the consistency of our extension.

## 3. Consistency of the extended ADHM construction

### 3.1 The conjugation

As we discussed in the previous section, to formulate the extended ADHM construction a kind of conjugation is needed, which maps the superfields $\hat{\Delta}_\alpha$ and $\hat{v}$ to $\widetilde{\hat{\Delta}}^\alpha$ and $\widetilde{\hat{v}}$, respectively. In order to determine the full form of the connection superfield $\phi_\mu$, it is found that we need to consider such a kind of conjugation also for other quantities. This "conjugation" operation should map a chiral superfield in $\mathcal{N}=1$ theory to another chiral superfield. Thus the ‡-conjugation satisfies at least the following properties: It maps the undotted (dotted) spinor $\theta^\alpha$ ($\bar{\theta}_{\dot{\alpha}}$) to itself and the chiral coordinate $y^\mu$ is invariant under this conjugation. In addition, the "conjugation" of the quantities appearing in the purely bosonic ADHM construction should reduce to the ordinary complex (hermitian) conjugation.

A conjugation suitable for our purpose (which we denote as "‡") has been considered in the literature (for example, [16]). The ‡-conjugation can be characterized by the following properties:

$$A^{\ddagger\ddagger} = (-)^{|A|}A\,, \tag{3.1}$$

$$(y^\mu)^\ddagger = y^\mu\,, \tag{3.2}$$

$$B^\ddagger = B^\dagger\,, \tag{3.3}$$

where $A$ denotes any quantity appearing in this paper ($|A|$ denotes its Grassmann oddness) and $B$ appearing in the purely bosonic ADHM construction. In addition, we require that $\theta_\alpha$ and $\bar{\theta}_{\dot{\alpha}}$ satisfy the following "reality" conditions:

$$(\theta_\alpha)^\ddagger = \theta^\alpha\,, \qquad (\bar{\theta}_{\dot{\alpha}})^\ddagger = \bar{\theta}^{\dot{\alpha}}\,. \tag{3.4}$$

We also introduce for each quantity its counterpart in the "anti-holomorphic" sectors with respect to the ‡-conjugation, which we indicate by a tilde "$\sim$". The rules we have adopted are given in appendix B.[6]

We require the "reality" of the supercharges under the ‡-conjugation as well as the Grassmann coordinates:

$$(Q_\alpha)^\ddagger = Q^\alpha\,, \qquad (\bar{Q}_{\dot{\alpha}})^\ddagger = \bar{Q}^{\dot{\alpha}}\,, \tag{3.5}$$

Then we can check that the supersymmetry algebra is self-conjugate under the ‡-conjugation.

Moreover, with the use of the ‡-conjugation (and the above "reality" conditions), we can obtain the real gauge potential $v_\mu$ in the euclidean version of the $\mathcal{N}=1$ SYM theory. We can impose the "reality" condition on the superfield $V$ in (2.7) as a superfield equation:

$$V^\ddagger = -V\,. \tag{3.6}$$

This condition implies[7]

$$v_\mu^\dagger = v_\mu\,, \qquad (\lambda_\alpha)^\ddagger = \lambda^\alpha\,, \qquad (\bar{\lambda}_{\dot{\alpha}})^\ddagger = \bar{\lambda}^{\dot{\alpha}}\,, \qquad D^\ddagger = -D\,. \tag{3.7}$$

Note that in our convention we have $(\theta\theta)^\ddagger = \theta\theta$, $(\bar{\theta}\bar{\theta})^\ddagger = \bar{\theta}\bar{\theta}$ and $(\theta\sigma^\mu\bar{\theta})^\ddagger = -\theta\sigma^\mu\bar{\theta}$. In the WZ gauge, as we have described in section 2.1, the gauge transformation (2.6) reduces to

$$e^V \mapsto e^{-i\varphi(\bar{y})}e^V e^{i\varphi(y)}\,, \tag{3.8}$$

where $\varphi$ is a complex valued function. After the imposition of (3.6), we find that $\varphi(y)^\ddagger = \varphi(y)$ is required. This leads to $\varphi(x)^\dagger = \varphi(x)$ and the condition $v_\mu^\dagger = v_\mu$ is preserved even after the gauge transformation. Thus we conclude that the gauge potential $v_\mu$ is real and the gauge group is SU($n$) or U($n$) rather than their complexification.

[6] In particular, the ‡-conjugation of a product of two fermionic quantities $A, B$ is defined by $(AB)^\ddagger = -B^\ddagger A^\ddagger$.

[7] The auxiliary field $D$ is hermitian in the lorentzian theory. On the other hand, according to the rule in appendix B, $D^\ddagger = -D$ implies that $D$ is anti-hermitian. Since we are considering the euclidean theory, the hermiticity of $D$ is not necessary.



JHEP08(2005)065

Also, the connection $\phi$ and the curvature $F$ have the following gauge freedom

$$\phi \mapsto X^{-1}\phi X - X^{-1}dX\,, \qquad F \mapsto X^{-1}FX\,, \tag{3.9}$$

where $X$ is a generic superfield extension of the ordinary gauge transformation. In our choice of the solution (2.2) to the covariant constraints, $(\phi_\mu)^\ddagger = -\phi_\mu$ holds, which follows from eq. (3.7) and can be checked with the use of the expression (2.10) in the WZ gauge. Therefore, the superfield $X$ should satisfies

$$X^\ddagger X = XX^\ddagger = \mathbf{1}_n \tag{3.10}$$

or $X^\ddagger = X^{-1}$ after imposition of (3.6).

Now let us apply the $\ddagger$-conjugation to the extended ADHM construction introduced in the previous section. According to the $\ddagger$-conjugation rules, we can check that $(\Delta_\alpha)^\ddagger$ coincides with $\widetilde{\hat{\Delta}}^\alpha$ defined in (2.29):

$$(\hat{\Delta}_\alpha)^\ddagger = \widetilde{\hat{\Delta}}^\alpha\,. \tag{3.11}$$

Also, $\widetilde{\hat{v}}$ appearing in (2.31) should also be identified with the "anti-holomorphic counterpart" of $\hat{v}$ with respect to the $\ddagger$-conjugation; from eqs. (2.33) and (2.34) we find that in the extended ADHM construction the $X$ transformation (3.9) is realized when $\hat{v}$ and its counterpart $\widetilde{\hat{v}}$ appearing in (2.31) transform as

$$\hat{v} \mapsto \hat{v}' = \hat{v}X\,, \qquad \widetilde{\hat{v}} \mapsto \widetilde{\hat{v}}' = X^\ddagger\widetilde{\hat{v}}\,, \tag{3.12}$$

where $X^\ddagger = X^{-1}$. This leads us to require $\widetilde{\hat{v}}$ to be the $\ddagger$-conjugate of $\hat{v}$:

$$\hat{v}^\ddagger = \widetilde{\hat{v}}\,. \tag{3.13}$$

According to these identifications, the $\ddagger$-conjugation rules tells us that the curvature superfield $F_{\mu\nu}$ in (2.40) given by the extended ADHM construction enjoys the following property:

$$(F_{\mu\nu})^\ddagger = -F_{\mu\nu}\,. \tag{3.14}$$

Here we have used $\hat{f}^\ddagger = \hat{f}$ which follows from (2.39). Since the lowest component of $F_{\mu\nu}$ is $-\frac{i}{2}v_{\mu\nu}$, the above equation tells us that

$$v_{\mu\nu}^\dagger = v_{\mu\nu}\,, \tag{3.15}$$

which further implies $v_\mu^\dagger = v_\mu$. This means that the gauge potential $v_\mu$ given by the extended ADHM construction can be kept and shown to be real if we adopt the $\ddagger$-conjugation.

### 3.2 The Wess-Zumino gauge and the component fields

In this subsection, we show that the extended ADHM construction and the $\ddagger$-conjugation can give the correct super instanton configurations. As we have discussed in the previous subsection, $\widetilde{\hat{\Delta}}^\alpha$ and $\widetilde{\hat{v}}$ in section 2.2 are identified with the "anti-holomorphic" counterpart of $\hat{\Delta}_\alpha$ and $\hat{v}$ with respect to the $\ddagger$-conjugation, respectively. Keeping this identification in mind, we can use all the equations in section 2.2 without any changes. We derive the normalized zero modes $\hat{v}$ of $\hat{\Delta}_\alpha$ by solving eq. (2.30) and determine the connection $\phi_\mu$.



JHEP08(2005)065

The zero mode $\hat{v}$ of $\hat{\Delta}_\alpha$ could be a general superfield. Given such a zero mode $\hat{v}$, we can find another gauge equivalent zero mode $\hat{v}'$ by the transformation (3.12). We can always expand $\hat{v}$ in terms of $\bar{\theta}$ as

$$\hat{v}(y,\theta,\bar{\theta}) = \hat{u}(y,\theta) + \bar{\theta}_{\dot{\alpha}}\hat{u}'^{\dot{\alpha}}(y,\theta) + \bar{\theta}\bar{\theta}\hat{u}''(y,\theta)\,, \tag{3.16}$$

where $\hat{v}(y,\theta,\bar{\theta})$ denotes the expression of the zero modes in the chiral basis. $\hat{u}$ should be normalized as $\widetilde{\hat{u}}\hat{u} = \mathbf{1}_n$ because of (2.32). Because $\hat{\Delta}$ is a chiral superfield, $\hat{\Delta}_\alpha\hat{v}$ is expanded in terms of $\bar{\theta}$ as

$$\hat{\Delta}_\alpha\hat{v}(y,\theta,\bar{\theta}) = \hat{\Delta}_\alpha\hat{u}(y,\theta) + \bar{\theta}_{\dot{\alpha}}\hat{\Delta}_\alpha\hat{u}'^{\dot{\alpha}}(y,\theta) + \bar{\theta}\bar{\theta}\hat{\Delta}_\alpha\hat{u}''(y,\theta) \tag{3.17}$$

and the zero mode equation leads to the following equations:

$$\hat{\Delta}_\alpha\hat{u}(y,\theta) = 0\,, \qquad \hat{\Delta}_\alpha\hat{u}'^{\dot{\alpha}}(y,\theta) = 0\,, \qquad \hat{\Delta}_\alpha\hat{u}''(y,\theta) = 0\,. \tag{3.18}$$

Once the normalized chiral superfield $\hat{u}$ is found, the fields $\hat{u}'^{\dot{\alpha}}$ and $\hat{u}''^{\dot{\alpha}}$ can be obtained with the use of $\hat{u}$ by

$$\hat{u}'^{\dot{\alpha}} = \hat{u}A^{\dot{\alpha}}\,, \qquad \hat{u}'' = \hat{u}B\,, \tag{3.19}$$

where $A^{\dot{1}}$, $A^{\dot{2}}$ and $B$ are arbitrary $n \times n$ matrices of chiral superfields. Therefore the zero mode $\hat{v}$ can always be written as

$$\hat{v}(y,\theta,\bar{\theta}) = \hat{u}(y,\theta)\mathbf{A}(y,\theta,\bar{\theta}) \tag{3.20}$$

where

$$\mathbf{A}(y,\theta,\bar{\theta}) \equiv \mathbf{1}_n(y,\theta) + \bar{\theta}_{\dot{\alpha}}A^{\dot{\alpha}}(y,\theta) + \bar{\theta}\bar{\theta}B(y,\theta)\,, \tag{3.21}$$

and $\mathbf{A}^{\ddagger} = \mathbf{A}^{-1}$ from the normalization conditions of $\hat{v}$ and $\hat{u}$. The superfield $\mathbf{A}$ is eliminated by the $X$ gauge transformation (3.12) with $X = \mathbf{A}^{-1}$. Therefore, we can always restrict the zero mode superfield $\hat{v}$ to a chiral superfield.

When $\hat{v}$ and $\widetilde{\hat{v}}$ are chiral superfields, we find from (2.33) that $\phi_{\dot{\alpha}} = 0$, and also that $\phi_\mu$ is a chiral superfield ($\bar{D}_{\dot{\alpha}}\phi_\mu = 0$) because eq. (2.33) tells us that $\phi_\mu$ is given by

$$\phi_\mu(y,\theta) = -\widetilde{\hat{v}}\frac{\partial}{\partial y^\mu}\hat{v}(y,\theta)\,. \tag{3.22}$$

These facts are consistent with the connection superfields for the super instantons (2.27). In the following, we show that we can actually find $\hat{v}$ which is a chiral superfield and correctly gives the connection superfields (2.27).

Considering $\phi_\alpha$, we find non-trivial necessary conditions to determine such $\hat{v}$. In the ADHM construction, $\phi_\alpha$ can be written in the chiral basis as

$$\phi_\alpha = -\widetilde{\hat{v}}D_\alpha\hat{v} = -\widetilde{\hat{v}}\frac{\partial}{\partial\theta^\alpha}\hat{v}(y,\theta) + 2i(\sigma^\mu\bar{\theta})_\alpha\phi_\mu(y,\theta)\,. \tag{3.23}$$

On the other hand, from (2.27) we see that $\phi_\alpha$ for the super instanton solutions should satisfy

$$\phi_\alpha = 2i(\sigma^\mu\bar{\theta})_\alpha\phi_\mu(y,\theta)\,, \tag{3.24}$$

in the WZ gauge. As a result, it should hold that

$$\widetilde{\hat{v}}\frac{\partial}{\partial\theta^\alpha}\hat{v}(y,\theta)=0\,, \tag{3.25}$$

which is a necessary condition for $\hat{v}$ to be in the WZ gauge. In the component language, denoting the zero mode chiral superfield $\hat{v}$ as

$$\hat{v}(y,\theta)\equiv v^{(0)}(y)+\theta^\gamma v^{(1)}_\gamma(y)+\theta\theta v^{(2)}(y)\,, \tag{3.26}$$

the above condition gives

$$\begin{aligned}\widetilde{v}^{(0)}v^{(1)}_\alpha &= 0\,, &(3.27)\\ \widetilde{v}^{(0)}v^{(2)}\delta^\beta{}_\alpha &= \frac{1}{2}\widetilde{v}^{(1)\beta}v^{(1)}_\alpha\,, &(3.28)\\ \widetilde{v}^{(1)}_\alpha v^{(2)} &= \widetilde{v}^{(2)}v^{(1)}_\alpha\,. &(3.29)\end{aligned}$$

In addition to this, the $\theta\theta$-component of the ASD connection $\phi_\mu$ should vanish in the WZ gauge:

$$\phi_\mu|_{\theta\theta}=\widetilde{v}^{(2)}\partial_\mu v^{(0)}+\widetilde{v}^{(0)}\partial_\mu v^{(2)}-\frac{1}{2}\widetilde{v}^{(1)\gamma}\partial_\mu v^{(1)}_\gamma=0\,. \tag{3.30}$$

We can use these conditions to determine $\hat{v}$ in the WZ gauge.

The zero dimensional Dirac equation (2.30) can be written as

$$\Delta_\alpha v^{(0)}+\theta^\gamma\Big(\Delta_\alpha v^{(1)}_\gamma+\varepsilon_{\alpha\gamma}\mathcal{M}v^{(0)}\Big)+\theta\theta\Big(\Delta_\alpha v^{(2)}+\frac{1}{2}\mathcal{M}v^{(1)}_\alpha\Big)=0\,. \tag{3.31}$$

With a given $v^{(0)}$ that satisfies $\Delta_\alpha v^{(0)}=0$ and $\widetilde{v}^{(0)}v^{(0)}=\mathbf{1}_n$, this equation is solved by

$$\begin{aligned}v^{(1)\gamma} &= \widetilde{\Delta}^\gamma f\mathcal{M}v^{(0)}+v^{(0)}\chi^\gamma\,, &(3.32)\\ v^{(2)} &= -\frac{1}{2}\widetilde{\Delta}^\gamma f\mathcal{M}\Big(\widetilde{\Delta}_\gamma f\mathcal{M}v^{(0)}+v^{(0)}\chi_\gamma\Big)+v^{(0)}s\,, &(3.33)\end{aligned}$$

where $\chi^\alpha$ and $s$ are an arbitrary $n\times n$ fermionic and bosonic matrix, respectively. With the use of the ‡-conjugation rules, we also find

$$\widetilde{\hat{v}}(y,\theta)=\widetilde{v}^{(0)}(y)+\theta^\gamma\widetilde{v}^{(1)}_\gamma(y)+\theta\theta\widetilde{v}^{(2)}(y)\,, \tag{3.34}$$

where $\widetilde{v}^{(0)}$ satisfies $\widetilde{v}^{(0)}\tilde{\Delta}^\alpha=0$ and $\widetilde{v}^{(0)}v^{(0)}=\mathbf{1}_n$, and

$$\begin{aligned}\widetilde{v}^{(1)}_\gamma &= -\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\Delta_\gamma+\widetilde{\chi}_\gamma\widetilde{v}^{(0)}\,, &(3.35)\\ \widetilde{v}^{(2)} &= \frac{1}{2}\Big(-\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\Delta^\gamma+\widetilde{\chi}^\gamma\widetilde{v}^{(0)}\Big)\widetilde{\mathcal{M}}f\Delta_\gamma+\widetilde{s}\widetilde{v}^{(0)}\,. &(3.36)\end{aligned}$$

To obtain this expression, we have used $(\widetilde{\Delta}^\gamma)^\ddagger=\Delta_\gamma$, $f^\ddagger=f$. We can check that $\widetilde{\hat{v}}$ given by the above components actually satisfies $\widetilde{\hat{v}}\widetilde{\hat{\Delta}}^\alpha=0$.

Next we will determine $\hat{v}$ in the WZ gauge. Substituting (3.32) and (3.33) into the necessary conditions (3.27) and (3.28), we find

$$\chi_\alpha=0\,,\qquad s=\frac{1}{2}\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\mathcal{M}v^{(0)}\,. \tag{3.37}$$

The third condition (3.29) is fulfilled by these $\chi_\alpha$ and $s$. The resulting zero mode $\hat{v}$ in the WZ gauge is

$$\hat{v} = v^{(0)} + \theta^\gamma\left(\widetilde{\Delta}_\gamma f\mathcal{M}v^{(0)}\right) + \theta\theta\left(\frac{1}{2}\widetilde{\mathcal{M}}f\mathcal{M}v^{(0)}\right), \tag{3.38}$$

and its conjugate is

$$\widetilde{\hat{v}} = \widetilde{v}^{(0)} + \theta^\gamma\left(-\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\Delta_\gamma\right) + \theta\theta\left(\frac{1}{2}\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\mathcal{M}\right). \tag{3.39}$$

Here we have used the fermionic ADHM constraint $\mathcal{M}\widetilde{\Delta}_\gamma = -\Delta_\gamma\widetilde{\mathcal{M}}$. We can check that this $\hat{v}$ satisfies the normalization condition.

We can explain the relation between the zero mode $\hat{v}$ given by (3.32) and (3.33), and the one in the WZ gauge (3.38), in terms of the $X$ gauge transformation. Note that even if the zero mode superfield $\hat{v}$ is taken to be a chiral superfield, there is a remaining $X$ gauge freedom (3.12) by a chiral superfield $X(y,\theta)$. Denoting the chiral superfield $X$ as

$$X(y,\theta) = X^{(0)}(y) + \theta^\gamma X^{(1)}_\gamma(y) + \theta\theta X^{(2)}(y), \tag{3.40}$$

we find that the components of $\hat{v}$ transform as

$$v^{(0)} \mapsto v^{(0)}X^{(0)}, \tag{3.41}$$

$$v^{(1)}_\alpha \mapsto v^{(1)}_\alpha X^{(0)} + v^{(0)}X^{(1)}_\alpha, \tag{3.42}$$

$$v^{(2)} \mapsto v^{(2)}X^{(0)} + v^{(0)}X^{(2)} - \frac{1}{2}v^{(1)\gamma}X^{(1)}_\gamma. \tag{3.43}$$

It is easy to check that the gauge transformation of the former $\hat{v}$ by the following $X$ gives $\hat{v}$ in the WZ gauge (3.38):

$$X = \mathbf{1}_n + \theta^\gamma\left(-\chi_\gamma\right) + \theta\theta\left(-s - \frac{1}{2}\chi^\gamma\chi_\gamma + \frac{1}{2}\widetilde{v}^{(0)}\widetilde{\mathcal{M}}f\mathcal{M}v^{(0)}\right). \tag{3.44}$$

Fixing the gauge to the one in which $\hat{v}$ is given as in (3.38), the $X$ symmetry breaks down to the ordinary SU($n$) (or U($n$)) gauge symmetry; now only $X(y,\theta) = X^{(0)}(y)$ is allowed, thus $X^\ddagger = X^{-1}$ means $X^{(0)}(y)^\ddagger = X^{(0)}(y)^{-1}$, which further implies that $X^{(0)}(x)^\dagger = X^{(0)}(x)^{-1}$.

We check that the above $\hat{v}$ correctly gives the connection $\phi$ in the WZ gauge of our solution (2.2). For the instanton solutions in the WZ gauge, $\phi_\mu$ should be $\phi_\mu = -\frac{i}{2}(v_\mu + i\theta\sigma_\mu\bar{\lambda})(y)$ with $v_\mu$ and $\bar{\lambda}$ being (2.17) and (2.22) respectively. In fact, substituting the above $\hat{v}$ and $\widetilde{\hat{v}}$ into eq. (2.33), we find that the connection now becomes

$$\phi_\mu = -\widetilde{v}^{(0)}\partial_\mu v^{(0)} + i\theta^\gamma\sigma_{\mu\gamma\dot{\beta}}\widetilde{v}^{(0)}(-\widetilde{b}^{\dot{\beta}}f\mathcal{M} - \widetilde{\mathcal{M}}fb^{\dot{\beta}})v^{(0)}. \tag{3.45}$$

Here the $\theta\theta$-component vanishes due to the relation

$$\partial_\mu f = f\Delta^\gamma\partial_\mu\widetilde{\Delta}_\gamma f \tag{3.46}$$

which can be proved with the use of the bosonic ADHM constraints and the canonical forms of $b$ and $\widetilde{b}$. Writing $v^{(0)} = v$, we see that $\widetilde{v}^{(0)} = v^\dagger$, $\widetilde{\mathcal{M}} = \mathcal{M}^\dagger$ and $\widetilde{b} = -b^\dagger$ follow from the definition of the ‡-conjugation (see appendix B), and the above $\phi_\mu$ coincides with the connection in the WZ gauge:

$$\phi_\mu = -\frac{i}{2}\left[-2iv^\dagger\partial_\mu v + i\theta^\gamma\sigma_{\mu\gamma\dot\beta}\left\{2iv^\dagger(b^{\dagger\dot\beta}f\mathcal{M} - \mathcal{M}^\dagger f b^{\dot\beta})v\right\}\right]. \tag{3.47}$$

Thus $\phi_\mu$ correctly reproduces the gauge field (2.17) and the fermion zero mode (2.22) of super instantons. Note that $\phi_\alpha$ satisfies (3.24) and $\phi_{\dot\alpha} = 0$, by construction.

With the use of (3.38), we can also check that $F_{\mu\nu}$ in (2.40) gives the correct component fields of (2.26). To verify this, we need $\hat f$: ¿From the eqs. (2.28) and (2.38), $\hat f^{-1}$ is given by

$$\hat f^{-1} = f^{-1} + \theta^\gamma\Delta_\gamma\widetilde{\mathcal{M}} + \frac{1}{2}\theta\theta\mathcal{M}\widetilde{\mathcal{M}} \tag{3.48}$$

and its inverse $\hat f$ is found to be

$$\hat f = f\left[f^{-1} - \theta^\gamma\Delta_\gamma\widetilde{\mathcal{M}} - \frac{1}{2}\theta\theta\mathcal{M}(1-\widetilde{\Delta}^\gamma f\Delta_\gamma)\widetilde{\mathcal{M}}\right]f\,. \tag{3.49}$$

Finally, let us discuss the dimension of the moduli space in the super field formalism of the instanton solution. The curvature $F$ is invariant under the following $\mathrm{GL}(k)\times\mathrm{U}(n+2k)$ global symmetry transformation

$$\hat\Delta_\alpha \to G\hat\Delta_\alpha\Lambda\,, \qquad \hat f \to G\hat f G^\dagger\,, \qquad \hat v \to \Lambda^{-1}\hat v\,, \tag{3.50}$$

where $G \in \mathrm{GL}(k)$ and $\Lambda \in \mathrm{U}(n+2k)$. After fixing $b$ in the canonical form, the global symmetry breaks down to $\mathrm{U}(n)\times\mathrm{U}(k)$ as in the purely bosonic ADHM construction. Therefore, the number of the bosonic parameters contained in $\hat\Delta_\alpha$ is $4nk$ as the usual bosonic case after imposing the bosonic ADHM constraints (2.19). The number of fermionic parameters contained in $\mathcal{M}$ is $2k(n+2k)$. There is no additional symmetry and thus the number of fermionic parameters is reduced simply by the fermionic ADHM constraints (2.23) and is $2kn$.

## 4. Conclusions and discussion

In this paper, we have studied the $\mathcal{N}=1$ superfield extension of the ADHM construction. We found a procedure to guarantee the WZ gauge for the superfields from the extended ADHM construction. Using the ‡-conjugation which enables us to impose a kind of reality condition, we have shown that the extended ADHM construction correctly gives super instanton configurations for the theory with $\mathrm{SU}(n)$ or $\mathrm{U}(n)$ gauge group, rather than their complexification.

It would be interesting to apply the extended ADHM construction to field theories in non(anti)commutative superspace [12]. The deformed super instanton configurations in such theories [23] can be obtained by replacing the product of superfields appearing in the extended ADHM construction by the Moyal type $*$-product. In order to find the component fields of super instantons, to guarantee the WZ gauge is especially important, and this step can be accomplished by using the ‡-conjugation. A detailed analysis will appear in a forthcoming paper [24].

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## A. Notation and conventions

By Wick rotation $x^0_{\rm E} = x_{\rm E0} = ix^0$, we obtain the euclidean theory with the metric $\eta^{\mu\nu} = \eta_{\mu\nu} = {\rm diag}(1,1,1,1)$ where $\mu,\nu = 0,1,2,3$. We drop the subscript "$_{\rm E}$".

The antisymmetric $\varepsilon$-tensors for the spinor indices are defined by

$$\varepsilon_{21} = \varepsilon^{12} = \varepsilon_{\dot{2}\dot{1}} = \varepsilon^{\dot{1}\dot{2}} = +1\,. \tag{A.1}$$

We use the following sigma matrices:

$$\sigma^\mu = \sigma_\mu \equiv (-i\mathbf{1}, \sigma^i)\,, \qquad \bar\sigma^\mu = \bar\sigma_\mu \equiv (-i\mathbf{1}, -\sigma^i)\,, \tag{A.2}$$

where $\sigma^i$ are the Pauli matrices and $\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$ holds. The Lorentz generators are

$$\sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \qquad \bar\sigma_{\mu\nu} \equiv \frac{1}{4}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)\,, \tag{A.3}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}\,, \qquad \bar\sigma^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\bar\sigma_{\lambda\rho}\,, \qquad \varepsilon^{0123} = \varepsilon_{0123} \equiv -1\,. \tag{A.4}$$

We define[8]

$$x_{\alpha\dot\beta} \equiv ix_\mu\sigma^\mu_{\alpha\dot\beta}\,, \qquad x^{\dot\beta\alpha} \equiv ix_\mu\bar\sigma^{\dot\beta\alpha}_\mu\,. \tag{A.5}$$

The supersymmetry algebra is

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu\,, \tag{A.6}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0\,, \qquad [P_\mu, \cdot\,] = 0\,. \tag{A.7}$$

In the coordinate representation, the generators are written as

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}\,, \qquad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,, \qquad P_\mu = i\frac{\partial}{\partial x^\mu}\,. \tag{A.8}$$

The supercovariant derivatives are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}\,, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,. \tag{A.9}$$

[8]Note that here the factor $i$ is contained so that these definitions to be the same as those using the quaternion basis in the spinor notation. However we do not use a bar "$^-$" to indicate the conjugation of a quaternion, since $\bar y_\mu$ is used to denote the antichiral coordinates.

The chiral and antichiral coordinates are

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,, \qquad \bar y^\mu = x^\mu - i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,. \tag{A.10}$$

We also define

$$y_{\alpha\dot\alpha} \equiv i y_\mu \sigma^\mu_{\alpha\dot\alpha}\,, \qquad y^{\dot\alpha\alpha} \equiv i y_\mu \bar\sigma^{\dot\alpha\alpha}_\mu\,. \tag{A.11}$$

Define the following basis in the cotangent space of the superspace:

$$(e^A) = (e^\mu, e^\alpha, e_{\dot\alpha})\,, \qquad e^\mu = dy^\mu - 2id\theta\sigma^\mu\bar\theta\,, \qquad e^\alpha = d\theta^\alpha\,, \qquad e_{\dot\alpha} = d\bar\theta_{\dot\alpha}\,. \tag{A.12}$$

Then the exterior derivative $d$ is expanded in the $\{e^A\}_{A=\mu,\alpha,\dot\alpha}$-basis as

$$d = e^A D_A\,, \tag{A.13}$$

where

$$(D_A) = \left(\frac{\partial}{\partial y^\mu}, D_\alpha, D^{\dot\alpha}\right). \tag{A.14}$$

The connection is a Lie algebra valued one-form:

$$\phi = e^A\phi_A\,, \qquad \phi_A = \phi^r_A iT^r\,, \tag{A.15}$$

($T^r$ are the hermitian generators in a certain representation of the gauge group). We will define

$$\phi_\mu|_{\theta=\bar\theta=0} \equiv -\frac{i}{2}v_\mu\,, \tag{A.16}$$

where $v_\mu = v^r_\mu T^r$ is the gauge potential field.

The curvature two-form $F$ is given by

$$F = d\phi + \phi\wedge\phi = \frac{1}{2}e^A\wedge e^B F_{BA}\,. \tag{A.17}$$

We find

$$F_{AB} = D_A\phi_B - (-)^{ab}D_B\phi_A - [\phi_A,\phi_B\} + T_{AB}{}^C\phi_C\,, \tag{A.18}$$

where $a$ and $b$ denote the Grassmann oddness of $\phi_A$ and $D_B$, respectively, $[\cdot,\cdot\}$ is a super-commutator and $T_{AB}{}^C$ is the torsion defined by $de^C = \frac{1}{2}e^Ae^BT_{BA}{}^C$ whose non-vanishing elements are $T_{\alpha\dot\beta}{}^\mu = T_{\dot\beta\alpha}{}^\mu = 2i\sigma^\mu_{\alpha\dot\beta}$. We can see

$$F_{\mu\nu}|_{\theta=\bar\theta=0} = -\frac{i}{2}v_{\mu\nu}\,, \tag{A.19}$$

where $v_{\mu\nu} = v^r_{\mu\nu}T^r$ is the usual field strength, $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + \frac{i}{2}[v_\mu, v_\nu]$.



JHEP08(2005)065

The curvature two-form $F$ satisfies the Bianchi identity: $\mathcal{D}F = dF - \phi\wedge F + F\wedge\phi = 0$, or

$$\frac{1}{2}e^A e^B e^C \left( D_C F_{BA} - [\phi_C, F_{BA}\} + \frac{1}{2}T_{CB}{}^D F_{DA} + \frac{1}{2}T_{CA}{}^D F_{DB} \right) = 0\,. \tag{A.20}$$

The "zero dimensional Dirac operator" in the extended ADHM construction is defined by

$$\hat{\Delta}_{\alpha[k]\times[n+2k]} = \Delta_\alpha + \theta_\alpha \mathcal{M}\,, \tag{A.21}$$

where

$$\Delta_\alpha \equiv a_\alpha + y_{\alpha\dot\beta} b^{\dot\beta} \tag{A.22}$$

and

$$\begin{aligned} a_{\alpha[k]\times[n+2k]} &\equiv \begin{pmatrix} \omega_{\alpha[k]\times[n]} & (a'_{\alpha\dot\beta})_{[k]\times[2k]} \end{pmatrix} = \begin{pmatrix} (\omega_\alpha{}^i{}_u) & (a'_{\alpha\dot 1}{}^i{}_j) & (a'_{\alpha\dot 2}{}^i{}_j) \end{pmatrix}, \\ \mathcal{M}_{[k]\times[n+2k]} &\equiv \begin{pmatrix} \mu_{[k]\times[n]} & (\mathcal{M}'_{\dot\beta})_{[k]\times[2k]} \end{pmatrix} = \begin{pmatrix} (\mu^i{}_u) & (\mathcal{M}'_{\dot 1}{}^i{}_j) & (\mathcal{M}'_{\dot 2}{}^i{}_j) \end{pmatrix}, \end{aligned} \tag{A.23}$$

with $u = 1,\ldots,n$ and $i,j = 1,\ldots,k$. Note that we write $a'_{\alpha\dot\beta}{}^j{}_i = i a'^{\mu j}{}_i \sigma_{\mu\alpha\dot\beta}$. The canonical form of $b$ is defined as

$$(b^{\dot\alpha}) = \begin{pmatrix} b^{\dot 1} \\ b^{\dot 2} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{[k]\times[n]} & \mathbf{1}_k & \mathbf{0}_k \\ \mathbf{0}_{[k]\times[n]} & \mathbf{0}_k & \mathbf{1}_k \end{pmatrix}. \tag{A.24}$$

## B. The ‡-conjugation rules

In this appendix, we will describe a map which we call the ‡-conjugation. There are many possibilities to choose a set of rules satisfying eqs. (3.1), (3.2) and (3.3) for imposing "reality" conditions like eq. (3.4). The rules described here are one of the possible ways which turns out to be sufficient for our purpose.

Let $A_{ij}$, $A^\alpha{}_{ij}$ and $\bar{A}^{\dot\alpha}{}_{ij}$ are matrices in the "holomorphic sector" in terms of the ‡-conjugation. We require the existence of the matrices in the "anti-holomorphic sector" $\widetilde{A}_{ij}$, $\widetilde{A}^\alpha{}_{ij}$ and $\widetilde{\bar{A}}^{\dot\alpha}{}_{ij}$, respectively. The matrices in the holomorphic sector and those in the anti-holomorphic sector are related by the ‡-conjugation as follows : For a (bosonic or fermionic) matrix $A$,

$$(A)^\ddagger = \widetilde{A}\,, \qquad (\widetilde{A})^\ddagger = (-)^{|A|}A\,. \tag{B.1}$$

For matrices with one spinor index,

$$\begin{aligned} (A_\alpha)^\ddagger &= \widetilde{A}^\alpha\,, \qquad (\widetilde{A}^\alpha)^\ddagger = (-)^{|A|}A_\alpha & \text{(B.2)} \\ (\bar{A}_{\dot\alpha})^\ddagger &= \widetilde{\bar{A}}^{\dot\alpha}\,, \qquad (\widetilde{\bar{A}}^{\dot\alpha})^\ddagger = (-)^{|\bar{A}|}\bar{A}_{\dot\alpha}\,. & \text{(B.3)} \end{aligned}$$

As a consequence, we have $A^{\ddagger\ddagger} = (-)^{|A|}A$, $(A^\alpha)^{\ddagger\ddagger} = (-)^{|A|}A^\alpha$ and $(\bar{A}^{\dot\alpha})^{\ddagger\ddagger} = (-)^{|\bar{A}|}\bar{A}^{\dot\alpha}$. The ‡-conjugation of a product of two (bosonic or fermionic) quantities is defined by

$$(AB)^\ddagger = (-)^{|A||B|}B^\ddagger A^\ddagger\,. \tag{B.4}$$

As long as the quantities in the anti-holomorphic sector are not specified, it is always possible to consider such a map for any quantity.

For the quantities appearing in the (non-supersymmetric) pure gauge theory and also in the ordinary ADHM construction, we require that the ‡-conjugation coincides with the usual hermitian (complex) conjugation †: For example,

$$v_\mu{}^\ddagger = v_\mu{}^\dagger \,, \qquad (\omega_\alpha)^\ddagger = (\omega_\alpha)^\dagger = \omega^{\dagger\alpha} \,, \qquad (b^{\dot\alpha})^\ddagger = (b^{\dot\alpha})^\dagger = b^\dagger{}_{\dot\alpha} \,, \tag{B.5}$$

$$(\varepsilon_{\alpha\beta})^\ddagger = (\varepsilon_{\alpha\beta})^* = -\varepsilon^{\alpha\beta} \,, \qquad (\sigma^\mu_{\alpha\dot\beta})^\ddagger = (\sigma^\mu_{\alpha\dot\beta})^* = -\bar\sigma^{\mu\dot\beta\alpha} \,, \qquad \text{etc.} \tag{B.6}$$

The changes of the upper and the lower spinor indices under the ‡-conjugation in (B.2) and (B.3) have been arranged to allow us this identification. Then the corresponding quantities in the anti-holomorphic sector are specified in accordance with the hermitian (complex) conjugation rule for each quantity in the holomorphic sector. In particular, we should notice that $\widetilde{b}_{\dot\alpha} = -b^\dagger_{\dot\alpha}$ in our convention.

For fermionic quantities with one spinor index, we are allowed to impose a "reality" condition in terms of the ‡-conjugation. This is accomplished by identifying the anti-holomorphic quantity with such a spinor itself. For example, we may require such spinors $\psi^\alpha$ and $\bar\psi^{\dot\alpha}$ to satisfy $\widetilde{\psi}^\alpha = \psi^\alpha$ and $\widetilde{\bar\psi}^{\dot\alpha} = \bar\psi^{\dot\alpha}$, that is,

$$(\psi_\alpha)^\ddagger = \psi^\alpha \,, \qquad (\bar\psi_{\dot\alpha})^\ddagger = \bar\psi^{\dot\alpha} \,. \tag{B.7}$$

The "reality" conditions (3.4) on $\theta$ and $\bar\theta$ are defined such that $y^\ddagger_\mu = y_\mu$ holds.

For the other quantities appearing in this paper, we identify the anti-holomorphic quantity with the hermitian (complex) conjugate, if it exists, of each holomorphic quantity like

$$\widetilde{\mathcal{M}} = \mathcal{M}^\dagger \,, \tag{B.8}$$

or with the quantity deduced from the known rules, if we can compute: For example, the anti-holomorphic counterpart of $\hat\Delta_\alpha$ can be computed by using the rules for $\Delta_\alpha$, $\theta_\alpha$ and $\mathcal{M}$, such that

$$\widetilde{\hat\Delta}^\alpha = \Delta^{\dagger\alpha} + \theta^\alpha \mathcal{M}^\dagger \,. \tag{B.9}$$

We should note that the hermitian conjugation and the ‡-conjugation for fermionic quantities do not commute in general. One of such examples is $\mathcal{M}$: We have $(\mathcal{M}^\ddagger)^\dagger = \mathcal{M}^{\dagger\dagger} = \mathcal{M}$, while $(\mathcal{M}^\dagger)^\ddagger = \widetilde{\mathcal{M}}^\ddagger = (-)^{|\mathcal{M}|}\mathcal{M}$.

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